SET THEORY HOMEWORK 4

Due Monday, November 4.

For the first two problems, let M be a countable transitive model of ZFC and let $Add(\omega, 1)$ be the poset of all partial finite functions from ω to $\{0, 1\}$. Let G be a generic filter over M.

Problem 1. *As we did in class, define $f^* : \omega \to \{0, 1\}$ to be $f^* = \bigcup G$. Recall that we proved in class that f^* is a total function with domain ω. Let $a = \{n \mid f^*(2n) = 0\}$.*

(1) *Show that both a and $\omega \setminus a$ are unbounded subsets of ω.*
(2) *Show that $a \notin M$.*

Problem 2. *Let $b \subset \omega$ be an infinite subset with $b \in M$. Define $\mathbb{P} = \{p \restriction b \mid p \in Add(\omega, 1)\}$ with the induced order: $p \leq q$ iff $p \supset q$. Show that $G_b := \{p \restriction b \mid p \in G\}$ is $\mathbb{P}$-generic over M.*

Problem 3. *Show that G is a $\mathbb{P}$-generic filter over M iff G is a filter such that G meets every maximal antichain $A \subset \mathbb{P}$ with $A \in M$.*

Problem 4. *Suppose that G is a $\mathbb{P}$-generic filter over M and $p \in G$.*

(1) *Suppose that $D \subset \mathbb{P}$ is* **dense below** *p i.e. for every $q \leq p$, there is $r \leq q$ with $r \in D$. Show that $G \cap D \neq \emptyset$.*
(2) *Let $A \subset \mathbb{P}$ be an antichain such that for every $q \in A, q \leq p$, and for every $r \leq p$, there is $q \in A$, such that r, q are compatible i.e. they have a common extension. Show that $G \cap A \neq \emptyset$. Such a set A is called a maximal antichain below p.*

Problem 5. *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a poset. Suppose that σ and τ are two $\mathbb{P}$-names in M, such that both* $\mathrm{dom}(\sigma), \mathrm{dom}(\tau) \subset \{\check{n} \mid n < \omega\}$.

(1) *Let*

$$\pi = \{\langle \check{n}, p\rangle \mid (\exists q, r)(p \leq q \wedge p \leq r \wedge \langle \check{n}, q\rangle \in \sigma \wedge \langle \check{n}, r\rangle \in \tau)\}.$$

Show that $\pi_G = \tau_G \cap \sigma_G$ for any generic filter G over M

(2) *Let*

$$\pi = \{\langle \check{n}, p\rangle \mid (\forall q \in \mathbb{P})(\langle \check{n}, q\rangle \in \sigma \to q \perp p)\}.$$

Show that $\pi_G = \omega \setminus \sigma_G$ for any generic filter G over M Hint: show that $\{r \mid \exists p \geq r(\langle \check{n}, p\rangle \in \pi \vee \langle \check{n}, p\rangle \in \sigma)\}$ is dense.